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Affine structures and KV-cohomology

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Abstract

We study transversely affine foliations with affine leaves from the point of view of Koszul–Vinberg modules. We have found a cohomological condition which assures that these structures determine an affine structure on the ambient manifold. The theoretical part is supplemented by suitable examples.

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1. Introduction

Affine manifolds play an important role in the theory of geometrical quantization—under some conditions a symplectic manifold carries also the structure of an affine manifold. To be precise, a careful look at the Konstant–Souriau quantization of classical observables discloses that it depends on a pair of supplementary Lagrangian subbundles. In his paper [2] Hess improved this classical method using bi-Lagrangian connections. His approach unifies several quantization methods. One of the important properties considered by Hess in his paper is the following theorem.

Theorem 1 (Hess). *Let $\mathcal{F}$ and $\mathcal{Q}$ be two supplementary Lagrangian subbundles of a symplectic manifold(M, ω). Then $\mathcal{F}$ and $\mathcal{Q}$ are Heisenberg related foliations iff the associated bi-Lagrangian connection is flat.*

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The Hess theorem assures that the symplectic manifold in question has an affine structure defined by a symplectic connection who induces canonical affine structures on leaves of both Lagrangian foliations.

Independently, the first author developed a similar theory of connections in view of an application to the theory of affine structures on nilpotent and solvable Lie groups, cf. [3,4].

In his paper [6] the second author took up the study of pairs of supplementary Lagrangian subbundles assuming that one of them is a foliation. A simple condition on the curvature of the associated bi-Lagrangian connection assured that the foliation was transversely affine. Namely, we have the following proposition.

Proposition 1. *Let $\mathcal{F}$ be a Lagrangian foliation on a symplectic manifold (M, ω). If $\mathcal{F}$ admits a supplementary Lagrangian subbundle for which the associated bi-Lagrangian connection is tangential, i.e. the mixed component of the curvature vanishes, then the foliation $\mathcal{F}$ is transversely affine.*

This proposition permitted to apply the theory of transversely affine foliations developed by the second author and prove some interesting properties of such pairs of supplementary Lagrangian subbundles, cf. [6,7].

In this short paper we propose to look at these structures from a different point of view. We assume that on a manifold we have a transversely affine foliation whose leaves are affine manifolds. We apply the theory of Koszul–Vinberg modules and their homologies developed by the first author [5] to obtain homological conditions which ensure that our manifold is affine and its flat connection induces the initial affine structures on leaves of the foliation as well as the initial transverse affine structure.

2. Notation and definitions

In this section we recall and introduce some useful notions and present chosen examples. All the objects considered are smooth, i.e. C^∞, unless otherwise stated.

c is a vector bundle $E \to M$ whose space of sections $\Gamma(E)$ (or more precisely the sheaf of sections) has the following properties:

(p_1) $\Gamma(E)$ is a Koszul–Vinberg algebra, i.e. to any two sections s, s' we associate a third one $s \cdot s'$ and this multiplication satisfies the following condition: for any three sections s, s' and s''

$$(s, s', s'') = (s', s, s''), \quad (s, s', s'') = s \cdot (s' \cdot s'') - (s \cdot s') \cdot s''.$$

(p_2) There exists a linear mapping (called anchor) $a : \Gamma(E) \to \mathcal{X}(M)$ satisfying the following for any $f \in C^\infty(M)$ and $s, s' \in \Gamma(E)$:

$$(fs) \cdot s' = f(s \cdot s'), \quad s \cdot (fs') = f(s \cdot s') + (\alpha(s)f) \cdot s'.$$

Definition 1. Given a KV-algebroid $E :\to M$ we denote by $\mathcal{L} = \mathcal{L}(E)$ the subset of $s_0 \in \Gamma(E)$ such that $ss_0 = 0$ for any $s \in \Gamma(E)$.

Elements of $\mathcal{L}$ are called linear sections E (or parallel vector fields). We also set $J(E) = \{\xi \in \Gamma(E) : (s, s', \xi) = 0 \text{ for any } s, s' \in \Gamma(E)\}$.

Remark. For any KV-algebra E the space of sections $\Gamma(E)$ is a Lie algebra with the bracket $[s, s'] = s \cdot s' - s' \cdot s$. Therefore, it follows from (p_2) that the anchor map is a morphism of the associated Lie algebras.

Example 1.

(i) If (M, D) is an affine manifold then the tangent bundle $E = TM$ is a KV-algebroid with the multiplication $s \cdot s' = D_s s'$ and the anchor mapping $a(s) = s$.

(ii) Let (M, ω) be a symplectic manifold. If $\mathcal{F}$ is a Lagrangian foliation on (M, ω) denote by E the subbundle of TM of vectors tangent to the leaves of the foliation $\mathcal{F}$. The KV-algebra multiplication in $\Gamma(E)$ is defined as follows:

$$i(s \cdot s')\omega = L_s i(s')\omega,$$

where L_s is the Lie derivation and $i(s)$ is the inner product.

One can easily check that the associated Lie algebra structure is the standard one.

(iii) For any manifold M its cotangent bundle T^*M carries the structure of a KV-algebroid associated to the vertical Lagrangian foliation of the standard symplectic manifold (T^*M, ω_0).

Next we would like to introduce an analogue of the normal bundle of a subbundle.

Definition 2. Let $E \to M$ be a KV-algebroid with an injective anchor a (such KV-algebroids are called regular). A KV-colagebroid of E is a vector fibre bundle $N \to M$ together with a linear application $\alpha : \Gamma(N) \to \Gamma(TM)$ such that

(P_1) $\Gamma(N)$ is a KV-algebra.

(P_2) There exists an exact sequence of Lie algebras

$$0 \to \Gamma(E) \overset{a}{\to} \Gamma(TM) \overset{j}{\to} \Gamma(N) \to 0. \tag{1}$$

(P_3) The anchor mapping $\alpha : \Gamma(N) \to \Gamma(TM)$ is a section of j and for any two sections s, s' of N and a smooth function f

- $(fs) \cdot s' = f(s \cdot s')$.
- $s \cdot (fs') = f(s \cdot s') + (\alpha(s)f)s'$ for any E-basic function f (i.e. $L_{\alpha(\sigma)} f = 0$ for any $\sigma \in \Gamma(E)$).

The mapping $\beta : \Gamma(TM) \to \Gamma(E)$ defined as $a(\beta(x)) = x - \alpha j(x)$ for any $x \in \Gamma(TM)$.

We recall that $J(N)$ is the subset of elements ξ of $\Gamma(N)$ such that $(s, s'\xi) = 0$ for any $s, s \in \Gamma(N)$. It is not difficult to notice that $J(N)$ is an associative subalgebra of the KV-algebra $\Gamma(N)$.

A manifold foliated by a transversely affine foliation $\mathcal{F}$ whose leaves carry affine structures provides us with the pair of a KV-algebroid and its coalgebroid. The subbundle $E(\mathcal{F})$ of vectors tangent to the leaves of the foliation is a KV-algebroid and the normal bundle $N(\mathcal{F})$ of the foliation is its coalgebroid. The anchor of the $E(\mathcal{F})$ is the standard inclusion and the anchor mapping of $N(\mathcal{F})$ is defined by a choice of the supplementary subbundle of the foliation (i.e. the identification of the normal bundle with a transverse subbundle to the foliation). In the paper of Benoit [1] one can find an example as above in which the manifold does not carry an affine structure (equivalently $\mathcal{X}(\mathcal{M})$ does not carry a KV-algebra structure) which induces the described algebraic structures on $E(\mathcal{F})$ and $N(\mathcal{F})$, i.e. the tangent and transverse affine structures. The aim of this note is to present an algebraic condition assuring that it is precise in this case.

3. *E*-parallel vector fields

Let $E \to M$ be a KV-algebroid. The sheaf $\mathcal{L}(E)$ is E-transitive, i.e. for sufficiently small open sets U the sections of $\mathcal{L}(U)$ generate $C^\infty(U)$-module $\Gamma(E)(U)$. We define the normalizer nor$(\mathcal{L})$ as follows:

$$\mathrm{nor}(\mathcal{L})(U) = \{X \in \Gamma(TM)(U) : [X, a(\mathcal{L}(U))] \subset a(\mathcal{L}(U))\}.$$

Next, we define a sheaf of Lie subalgebras

$$\mathcal{A}(E) = \mathrm{nor}(\mathcal{L})(M) \cap \alpha(\Gamma(N)).$$

We have an exact sequence of sheaves of vector spaces

$$0 \to \mathcal{L} \xrightarrow{a} \mathcal{A}(E) \xrightarrow{j} \mathcal{N} \to 0,$$

where $\mathcal{N} = j(\mathcal{A}(E))$.

Now for any $s \in \mathcal{N}$ and $l \in \mathcal{N}(\mathcal{L})$ we put

$$a(s \cdot l) = [v, a(l)],$$

and

$$l \cdot s = 0,$$

where $j(v) = s$. The definition is independent of the choice of v as if $s = j(v) = j(v')$ then $j(v - v') = 0$. So $v - v' = a(w)$ for some $w \in \mathcal{L}$. Thus $[(v - v'), a(l)] = [a(w), a(l)] = a[w, l] = 0$, hence $[v, a(l)] = [v', a(l)]$.

Let denote by $\mathcal{N}_L$ the vector space $\Gamma(\mathcal{N})$ considered as the Lie algebra with the bracket induced by the bracket in $\mathcal{A}(E)$.

$\mathcal{L}$ is a trivial KV-algebra sheaf. According to the above definition $\mathcal{L}$ is a trivial right KV-module of $\mathcal{N}$ and a left $\mathcal{N}_L$-module.

Let us define $\omega : \mathcal{N}_L \times \mathcal{N}_L \to \mathcal{E}$ to be the measure of the fact that the splitting α is not a Lie algebra homomorphism:

$$\omega(s, s') = [\alpha(s), \alpha(s')] - \alpha([s, s'])$$

for any sections s, s' of $\mathcal{N}$.

ω takes values in $\mathcal{N}(\mathcal{L})$ if the splitting α takes values in $\mathcal{A}(E)$.

Since the exact sequence

$$0 \to \mathcal{L} \xrightarrow{a} \mathcal{A}(E) \xrightarrow{j} \mathcal{N}_L \to 0$$

is a sequence of Lie algebras (with $[\mathcal{L}, \mathcal{L}] = 0$) ω is a two-cocycle of the Hochschild complex $C^*(\mathcal{N}_L, \mathcal{N}(\mathcal{L}))$.

Let $J(\mathcal{L}) = \{l \in \mathcal{L} : (s, s', l) = 0\}$ for any sections $s, s' \in \mathcal{N}$, where $(s, s', l) = s(s' \cdot l) - (s \cdot s') \cdot l$.

The KV-complex of $\mathcal{N}$ with coefficients in $\mathcal{N}(\mathcal{L})$ is the sequence

$$J(\mathcal{L}) \xrightarrow{\delta_0} C_1(\mathcal{N}, \mathcal{N}(\mathcal{L})) \xrightarrow{\delta_1} C_2(\mathcal{N}, \mathcal{N}(\mathcal{L})) \cdots ,$$

where the mapping $\delta_q : C_q(\mathcal{N}, \mathcal{N}(\mathcal{L})) \to C_{q+1}(\mathcal{N}, \mathcal{N}(\mathcal{L}))$ is defined as follows

$$q > 0, \quad C_q(\mathcal{N}, \mathcal{N}(\mathcal{L})) = \mathrm{Hom}(\times^q \mathcal{N}, \mathcal{N}(\mathcal{L})),$$

and

$$(\delta_0(l))(s) = -s \cdot l + l \cdot s,$$

$$\delta_q \theta(s_1, \ldots, s_{q+1}) = \sum_{p \le q} (-1)^p \{(s_p \theta)(\cdots \hat{s}_p \cdots s_{q+1}) + (e_q(s_p)\theta s_{q+1})(\cdots \hat{s}_p \cdots \hat{s}_{q+1})\},$$

where $(e_q(s_p)\theta s_{q+1})(\xi_1, \ldots, \xi_{q-1}) = \theta(\xi_1, \ldots, \xi_{q-1}, sq)s_{q+1}$.

Proposition 2. *The Koszul–Spencer operator induces a natural injection of* $H_2(\mathcal{N}, \mathcal{L})$ *into* $H^2(\mathcal{N}_L, \mathcal{L})$.

Proof. One sends $C_2(\mathcal{N}, \mathcal{L})$ into $C^2(\mathcal{N}_L, \mathcal{L})$ by the formula $\theta \mapsto \partial\theta$, where $(\partial\theta)(s, s') = \theta(s, s') - \theta(s', s)$. Simple calculations show that $\delta\theta = 0$ implies that $\mathrm{d}\partial\theta = 0$, thus $\mathrm{d} : C^2(\mathcal{N}_L, \mathcal{L}) \to C^3(\mathcal{N}_L, \mathcal{L})$ the coboundary operator of Chevalley–Hoschild.

On the other hand if θ is δ-exact, then $\theta(s, s') = -s \cdot \phi(s') + \phi(s \cdot s') - \phi(s) \cdot s'$, then one easily sees that $\partial\theta$ is d-exact. It is not difficult to verify that

$$\mathrm{d}\phi(s, s') = -\partial\delta\phi(s, s').$$

This last identity ensures that the mapping in question is injective. □

We put

$$\mathcal{O}(E, N) = \frac{H^2(N_L, \mathcal{N}(\mathcal{L}))}{H_2(N_L, \mathcal{N}(\mathcal{L}))}.$$

Thus we have an exact sequence

$$0 \to H_2(N_L, \mathcal{N}(\mathcal{L})) \to H^2(N_L, \mathcal{N}(\mathcal{L})) \to \mathcal{O}(E, N) \to 0.$$

The image of the cohomology class $[\omega] \in H^2(N_L, \mathcal{N}(\mathcal{L}))$ in $\mathcal{O}(E, N)$ is denoted by $\mathrm{o}(E, N)$.

Theorem 2. *Let* $0 \to E \to TM \to N \to 0$ *be an exact sequence of vector bundles. Assume that* $\mathcal{N}$ *generates* $\Gamma(N)$ *as a* $C^\infty(M, R)$*-module. Moreover, let E be a KV-algebroid and N* *be its KV-coalgebroid. If the right splitting α of the exact sequence takes values in $\mathcal{N}$ into* $\mathrm{nor}(\mathcal{L})$, *then the manifold M admits an affine structure compatible with the KV-structures on E and N iff* $\mathrm{o}(E, N) = 0$.

Proof of Theorem 1. Assume that the manifold M has an affine structure D which induces affine structures on the leaves of the foliation E, thus $\mathcal{E}$ is a subsheaf of the sheaf TM of KV algebras. In particular leaves of the foliation E are totally geodesic submanifolds for the connection D.

Our assumptions permit us to define a two-chain $\theta : \mathcal{N} \to \mathcal{L}$ by the following formula:

$$\theta(s, s') = \alpha(s) \cdot \alpha(s') - \alpha(s \cdot s') = D_{\alpha(s)}\alpha(s') - \alpha(s \cdot s').$$

We can identify $\mathcal{A}(E)(U)$ with $\mathcal{N}(\mathcal{L})(U) \oplus \mathcal{N}(U)$ and put

$$(l, s) \cdot (l', s') = s \cdot l' + \alpha(s \cdot s') + \theta(s, s').$$

Then θ is a two-cocycle. Since $[(l, s), (l', s')] = [l + \alpha(s), l' + \alpha(s')] = (l, s) \cdot (l', s') - (l', s') \cdot (l, s)$, we get that

$$\omega(s, s') = \partial\theta(s, s').$$

This last equality implies that

$$\mathrm{o}(E, \mathcal{N}) = 0.$$

Now assume that $\mathrm{o}(E, \mathcal{N}) = 0$. Then there exists a class $[\theta]$ in $H_2(\mathcal{N}, \mathcal{L})$ such that $\partial[\theta] = [\omega] \in H^2(\mathcal{N}_L, \mathcal{L})$. Let $\theta \in [\theta]$. One can choose $\theta \in [\theta]$ such that $\partial\theta = \omega$ and $\delta\theta = 0$. Then we put (in $\mathcal{A}(E)$) $s \cdot s' = \alpha(s) \cdot \alpha(s') = \alpha(s \cdot s') + \theta(s, s')$ and then

$$(l + s) \cdot (l' + s') = (l + \alpha(s)) \cdot (l' + \alpha(s')) = [\alpha(s), l'] + \theta(s, s') + \alpha(s \cdot s').$$

One can easily verify that

$$(l + s, l' + s', l'' + s'') = (l' + s', l + s, l'' + s'').$$

Our assumption that local sections of the sheaf $\mathcal{A}(E)$ span the tangent bundle TM that the multiplication defined above defines a linear connection, i.e.

$$(f(l + s)) \cdot (l' + s') = f((l + s) \cdot (l' + s')),$$

and

$$(l + s) \cdot (f(l' + s')) = f((l + s) \cdot (l' + s')) + ((l + \alpha(s))f)(l' + s').$$

It is not difficult to verify that the connection determined by the KV-multiplication satisfies the conditions of our theorem. □

4. Examples

Example 2. Let $M = R^2$. We denote by E the subbundle of *TM* generated by $\partial/\partial x$. Let fix $h \in C^\infty(M, R)$ and define the multiplication in $\Gamma(E)$ by the following formula:

$$f\frac{\partial}{\partial x}\cdot g\frac{\partial}{\partial x} = \left(f\frac{\partial g}{\partial x} + fg\frac{\partial h}{\partial x}\right)\frac{\partial}{\partial x}. \tag{2}$$

This multiplication defines a Koszul–Vinberg algebra. The subalgebra $\mathcal{L} = \mathcal{L}(E)$ consists of elements of the form $\lambda(y)\,\mathrm{e}^{-h(x,y)}\partial/\partial x$, where $\lambda \in C^\infty(R, R)$.

Let ξ be a vector field defined as follows:

$$\xi(x, y) = \left[\frac{\partial}{\partial y}\int_0^x \mathrm{e}^{h(t,y)}\,\mathrm{d}t\right]\mathrm{e}^{-h(x,y)}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Let N be the subbundle of *TM* generated by ξ. In $\Gamma(N)$ we define the multiplication as follows:

$$f\xi \cdot g\xi = f\xi(g)\xi, \tag{3}$$

where $\xi(g) = \langle \mathrm{d}g, \xi\rangle$. In this way we define a Koszul–Vinberg algebra structure in $\Gamma(N)$. Let $I(E)$ be the subalgebra of the associative algebra $C^\infty(M, R)$ consisting of first integrals of E; it is the "inverse" image of $C^\infty(R, R)$ by the canonical projection $p_2 : R \times R \to R$, $p_2(x, y) = y$. Thus $f \in I(E)$ iff $\partial f/\partial x = 0$. One can associate to any $\alpha \in I(E)C^\infty(M, R)$-linear mapping of N into *TM* defined by

$$\tilde{\alpha}(\xi)(x, y) = \left(\alpha(y) + \frac{\partial}{\partial y}\int_0^x \mathrm{e}^{h(t,y)}\,\mathrm{d}t\right)\mathrm{e}^{-h(x,y)}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}. \tag{4}$$

Now we can consider the couple $(N, \tilde{\alpha})$ as a coalgebroid of the Koszul–Vinberg algebroid E. We have

$$TM = E \oplus \tilde{\alpha}(N),$$

and the exact sequence

$$0 \to E \to TM \xrightarrow{j} N \to 0,$$

where

$$j\left(X(x, y)\frac{\partial}{\partial x}\right) + Y(x, y)\left(\alpha(y)\frac{\partial}{\partial y} + \xi(x, y)\right) = Y(x, y)\xi(x, y). \tag{5}$$

In the sequel we assume that $\alpha \neq 0$.

Lemma 1. *$\Gamma(N)$ contains a non-null subalgebra whose image by the anchor $\tilde{\alpha}$ is in* $\mathrm{nor}(\mathcal{L})$.

Proof. The subalgebra of vector fields $X\partial/\partial x + Y\partial/\partial y$ in the normalizer can be identified with couples $(X, Y) \in C^\infty(M, R) \times C^\infty(M, R)$ which constitute solutions of the following system of partial differential equations:

$$X(x, y)\frac{\partial h(x, y)}{\partial x} + \frac{\partial X(x, y)}{\partial x} + Y(x, y)\frac{\partial h(x, y)}{\partial y} = 0, \qquad \frac{\partial Y(x, y)}{\partial x} = 0.$$

Let $Y(x, y)\xi \in \Gamma(N)$, then $\tilde{\alpha}(Y\xi) \in \mathrm{nor}(\mathcal{L})$ iff $Y \in I(E)$, which is equivalent to $\partial Y/\partial x = 0$.

Let us denote by η the subalgebra of the Koszul–Vinberg algebra $\Gamma(N)$ consisting of vector fields of the form

$$Y(y)(\xi(x, y)) = Y(y)\left\{\left(\frac{\partial}{\partial y}\int_0^x \mathrm{e}^{h(t,y)}\,\mathrm{d}t\right)\mathrm{e}^{-h(x,y)}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right\}.$$

It is not difficult to verify that $\tilde{\alpha}(\eta) \subset \mathrm{nor}(\mathcal{L})$. □

Let

$$\mathcal{A} = \mathrm{nor}(\mathcal{L}) \cap \tilde{\alpha}(\eta).$$

It is a Lie subalgebra of the Lie algebra $\Gamma(TM)$, it is a transitive one—the $C^\infty(M, R)$-module generated by $\mathcal{A}$ is equal to $\Gamma(TM)$. Moreover, we have the following exact sequence

$$0 \to \mathcal{L} \to \mathcal{A} \to \eta \to 0$$

which is split by the anchor map $\tilde{\alpha}$. One can easily verify that with the standard bracket of vector fields in η:

$$[f\xi, g\xi] = [f\xi(g) - g\xi(f)]\xi.$$

The anchor is a Lie algebra homomorphism, i.e.

$$\tilde{\alpha}[f\xi, g\xi] = [\tilde{\alpha}(f\xi), \tilde{\alpha}(g\xi)],$$

and $\tilde{L}$ is an abelian subalgebra of $\Gamma(TM)$. The algebra $\mathcal{A}$ is a semi-direct product of η by $\mathcal{L}$.

One can consider $\mathcal{L}$ as a Koszul–Vinberg module of η by posing

$$\xi \cdot l = [\tilde{\alpha}(\xi), l], \qquad l \cdot \xi = 0$$

for any $l \in \mathcal{L}$. Then one can define in $\mathcal{A}$ a structure of a Koszul–Vinberg algebra such that the exact sequence

$$0 \to \mathcal{L} \to \mathcal{A} \xrightarrow{j} \eta \to 0$$

is a sequence of Koszul–Vinberg algebras.

In fact, due to the main theorem the cocycle associated to $\mathcal{A} = \mathcal{L} \times \eta$ is equal to 0. It can be represented as an exact cycle $\phi \in C_2(\eta, \mathcal{L})$. On the other hand, ϕ is of the form

$$\phi(f\xi, g\xi) = [\tilde{\alpha}(f\xi), \psi(g\xi)] - \psi(f\xi \cdot g\xi),$$

where $\psi \in \mathrm{Hom}_R(\eta, \mathcal{L})$.

Example 3. Let us consider the same E as in Example 2 with the same multiplication in $\Gamma(E)$. Take the subbundle N of TM spanned by $\partial/\partial y$. The space $\Gamma(N)$ of sections is equipped with multiplication

$$f\frac{\partial}{\partial y} \cdot g\frac{\partial}{\partial y} = f\left(\frac{\partial g}{\partial y}\right)\frac{\partial}{\partial y}.$$

The pair $(N, \text{identity})$ is a coalgebroid of E. Let $Y(x, y)\partial/\partial y \in \Gamma(N)$, then it is in $\text{nor}(\mathcal{L})$ iff

$$\frac{\partial Y}{\partial x}\frac{\mathrm{d}\lambda}{\mathrm{d}y} - \lambda(y)\frac{\partial^2 h}{\partial x \partial y} = 0$$

for any $\lambda \in I(E)$.

The above equation is equivalent to the following two equations:

$$\frac{\partial Y}{\partial x} = 0, \qquad \frac{\partial^2 h}{\partial x \partial y} = 0.$$

To assure that $\text{nor}(\mathcal{L}) \cap \Gamma(N)$ be transitive it is necessary that the function h be of the form

$$h(x, y) = a(x) + b(y),$$

where $a, b \in C^\infty(R, R)$. If not, $\Gamma(N)$ does not contain any Koszul–Vinberg subalgebra of $\text{nor}(\mathcal{L})$. From the above system one can easily deduce that in the general case the transverse structure of an affine foliation plays a crucial role as $I(E) \neq R$ excludes the existence of a dense leaf.

Example 4. Let $M = S^1 \times R^2$ with the coordinates θ, x, y. They define global commuting parallelism: $\partial/\partial\theta$, $\partial/\partial x$, $\partial/\partial y$. Let us consider the subbundle E of TM spanned by $\partial/\partial\theta$ and by N the subbundle spanned by $\partial/\partial x$ and $\partial/\partial y$.

In $\Gamma(E)$ we have the following multiplication:

$$f\frac{\partial}{\partial\theta} \cdot g\frac{\partial}{\partial\theta} = f\left(\frac{\partial g}{\partial\theta}\right)\frac{\partial}{\partial\theta}, \tag{6}$$

and in $\Gamma(N)$ the following:

$$\left(f_0\frac{\partial}{\partial x} + g_0\frac{\partial}{\partial y}\right)\left(f_1\frac{\partial}{\partial x} + g_1\frac{\partial}{\partial y}\right) = \left(f_0\frac{\partial f_1}{\partial x} + g_0\frac{\partial f_1}{\partial y} + f_0 f_1\right)\frac{\partial}{\partial x} + \left(f_0\frac{\partial g_1}{\partial x} + g_0\frac{\partial g_1}{\partial y} + g_0 g_1\right)\frac{\partial}{\partial y}. \tag{7}$$

With the above defined multiplications $\Gamma(E)$ and $\Gamma(N)$ are Koszul–Vinberg algebras. Consider the application α of $\Gamma(N)$ into $\Gamma(TM)$ given by the following formulae:

$$\alpha\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}, \qquad \alpha\left(\frac{\partial}{\partial y}\right) = x\frac{\partial}{\partial\theta} + \frac{\partial}{\partial y}.$$

The couple (N, α) is a Koszul–Vinberg coalgebroid of E. In fact for any $s, s' \in \Gamma(N)$ and any $f \in C^\infty(M, R)$ we have

$$s(fs') = fss' + \alpha(s)fs'$$

iff $f \in I(E)$.

On the other hand, due to (6), $\mathcal{L} = \mathcal{L}(E)$ is the subspace $I(E)\partial/\partial\theta$. From the very definition im α is in nor$(\mathcal{L})$. Let us consider the free $I(E)$-submodule $\mathcal{A}$ of $\Gamma(TM)$ generated by $\{\partial/\partial\theta, \alpha(\partial/\partial x), \alpha(\partial/\partial y)\}$ or by $\{\partial/\partial\theta, \partial/\partial x, \partial/\partial y + x\partial/\partial\theta\}$. It is a Lie subalgebra of $\Gamma(TM)$. One can obtain $\mathcal{A}$ as a semi-direct product of $I(E)\partial/\partial x + I(E)\partial/\partial y$ by $I(E)\partial/\partial\theta$. $\mathcal{A}$ contains as a subalgebra the Heisenberg algebra

$$\mathcal{H}_{\mathcal{Z}} = \mathrm{span}_R \left\{ \frac{\partial}{\partial\theta}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + \frac{x\partial}{\partial\theta} \right\}$$

which is the semi-direct product of Vect$\{\partial/\partial x, \partial/\partial y\}$ by $R\partial/\partial\theta$. The cocycle ω of this extension:

$$R\frac{\partial}{\partial\theta} \to \mathcal{H}_{\mathcal{Z}} \to \mathrm{Vect}\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}$$

is the following

$$\omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) = \left[\alpha\left(\frac{\partial}{\partial x} \right), \alpha\left(\frac{\partial}{\partial y} \right) \right] = \frac{\partial}{\partial\theta}.$$

The ideal $\mathcal{L}_R = R\partial/\partial\theta$ of $\mathcal{H}_{\mathcal{Z}}$ inherits the null multiplication, but $\eta = \mathrm{Vect}\{\partial/\partial x, \partial/\partial y\}$ inherits the following multiplication from $\Gamma(N)$:

$$\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right) \cdot \left(a'\frac{\partial}{\partial x} + b'\frac{\partial}{\partial y} \right) = aa'\frac{\partial}{\partial x} + bb'\frac{\partial}{\partial y} \tag{8}$$

for any $a, a', b, b' \in R$.

We define the action of η on $\mathcal{L}_R$ by the following formula:

$$\begin{aligned} \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right) \cdot c\frac{\partial}{\partial\theta} &= \left[\alpha\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right), c\frac{\partial}{\partial\theta} \right] \\ &= \left[a\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x\frac{\partial}{\partial\theta}, c\frac{\partial}{\partial\theta} \right] = 0, \end{aligned} \tag{9}$$

and

$$c\frac{\partial}{\partial\theta}\left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right) = 0$$

for any $a, b, c \in R$.

Let $\Phi \in C_2(\eta, \mathcal{L}_R)$. With respect to the basis $\{\partial/\partial x, \partial/\partial y\}$ Φ is represented by the matrix

$$\begin{pmatrix} u & v \\ w & t \end{pmatrix}.$$

Taking into account (9) we have

$$\partial_2\Phi(X, Y, Z) = -\Phi(XY, Z) - \Phi(Y, XZ) + \Phi(YX, Z) + \Phi(X, YZ).$$

Then taking into account (8) we get

$$\partial_2\Phi(X, Y, Z) = -\Phi(Y, XZ) + \Phi(X, YZ).$$

Take $X = (a, b)$, $Y = (a', b')$, $Z = (a'', b'')$, then $XZ = (aa'', bb'')$ and $YZ = (a'a'', b'b'')$, and hence

$$\begin{aligned}\partial_2\Phi(X, Y, Z) &= -(a'b')\begin{pmatrix} u & v \\ w & t \end{pmatrix}\begin{pmatrix} aa'' \\ bb'' \end{pmatrix} + (ab)\begin{pmatrix} u & v \\ w & t \end{pmatrix}\begin{pmatrix} a'a'' \\ b'b'' \end{pmatrix} \\ &= -aa''(ua'+wb') - bb''(va' + tb') + a'a''(ua + wb) + b'b''(va + tb) \\ &= w(a'a''b - aa''b') + v(ab'b'' - a'bb'').\end{aligned}$$

Thus $\partial_2\Phi = 0$ iff $v = w = 0$, that is iff

$$\Phi = uaa' + tbb'.$$

Therefore, the mapping ∂ from $H_2(\eta, \mathcal{L}_R)$ to $H^2(\eta_L, \mathcal{L}_R)$ is zero. The obstruction $o([\omega]) \in \mathcal{O}(\eta, \eta_L)$ is not zero, thus the manifold M does not carry an affine structure inducing the structures (6) and (7).

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